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# EXTREMUM PROBLEMS OF BOUNDARY CONTROL FOR STEADY EQUATIONS OF THERMAL CONVECTION

**G. V. Alekseev and D. A. Tereshko**

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*An inverse extremum problem of boundary control for steady equations of thermal convection is considered. The cost functional in this problem is chosen to be the root-mean-square deviation of flow velocity or vorticity from the velocity or vorticity field given in a certain part of the flow domain; the control parameter is the heat flux through a part of the boundary. A theorem on sufficient conditions on initial data providing the existence, uniqueness, and stability of the solution is given. A numerical algorithm of solving this problem, based on Newton's method and on the finite element method of discretization of linear boundary-value problems, is proposed. Results of computational experiments on solving extremum problems, which confirm the efficiency of the method developed, are discussed.*

**Key words:** *thermal convection, extremum problems, uniqueness, stability, algorithm, Newton's method.*

**1. Formulation of the Boundary-Value Problem.** The theory of controlling thermohydrodynamic processes in liquid media is experiencing intense recent development. There are many publications on solving control problems and inverse extremum problems for steady models of heat and mass transfer (see, e.g., [1–8]), where solvability of extremum problems was proved and optimality systems that describe the necessary conditions of the extremum point were derived and studied. Based on an analysis of the optimality system, conditions of uniqueness and stability of solutions of control problems in particular cases corresponding to purely hydrodynamic or temperature cost functionals and controls were established in [5–8]. The issues of controlling viscous fluid flows with the use of heat sources are less studied, but the computational results show that it is a method that can reveal effective mechanisms of controlling viscous fluid flow regimes.

As an example, let us consider the classical problem of a viscous fluid flow around a cylinder in a channel. Vortices are known to form behind the cylinder in certain flow regimes, which increases the drag force acting on the body from the viscous fluid, as compared to the case of a non-separated flow. Computational experiments show, however, that it is possible to ensure a non-separated flow by choosing an appropriate regime of heating or cooling of the cylinder walls and closely located channel walls. In turn, this leads to significant reduction of the drag force, as compared to the uncontrolled flow case. To ensure such a regime, it is necessary to solve a corresponding control problem for the classical model of thermal convection with the heat flux through the body and channel walls chosen as a control function. The objective of the present work is to study similar control problems for the following model of thermal convection:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} + \operatorname{grad} p = (1-\beta_T T)\boldsymbol{G}, \qquad \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega; \tag{1.1}$$

$$-\lambda\Delta T + \boldsymbol{u}\cdot\operatorname{grad} T = 0 \qquad \text{in } \Omega; \tag{1.2}$$

$$\boldsymbol{u}\Big|_{\Gamma} = \boldsymbol{g}, \qquad T\Big|_{\Gamma_D} = \psi, \qquad \lambda\frac{\partial T}{\partial n}\Big|_{\Gamma_N} = \chi. \tag{1.3}$$

Institute of Applied Mathematics, Far-East Division, Russian Academy of Sciences, Vladivostok 690041; alekseev@iam.dvo.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 51, No. 4, pp. 72–84, July–August, 2010. Original article submitted February 5, 2009; revision submitted July 24, 2009.

 

Here $\Omega$ is the bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the boundary $\Gamma$ consisting of two parts ($\Gamma_D$ and $\Gamma_N$), $\boldsymbol{u}$ and $T$ are the velocity and temperature, respectively, $p = P/\rho$ ($P$ is the pressure and $\rho = \text{const}$ is the medium density), $\nu > 0$ and $\lambda > 0$ are the kinematic viscosity and thermal diffusivity, which are constants, $\boldsymbol{G}$ is the free-fall acceleration, and $\beta_T$, $\boldsymbol{g}$, $\psi$, and $\chi$ are certain functions.

The problem considered is aimed at finding the function $\chi$ in Eq. (1.3), which has the meaning of the heat flux through the boundary, and the solution $(\boldsymbol{u}, p, T)$ of problem (1.1)–(1.3), which satisfies the condition of the minimum of a certain velocity-dependent cost functional. A theorem on sufficient conditions that ensure uniqueness and stability of the solution of the corresponding extremum problem is proved, and a numerical algorithm of solving this problem, based on Newton's method, is proposed. Some results of computational experiments are discussed, which confirm the theoretically predicted possibility of effective control of the viscous flow regime by choosing appropriate boundary sources of heat.

Let us introduce the following notation. The scalar products in $L^2(\Omega)$, $L^2(Q)$ $(Q \subset \Omega)$, or $L^2(\Gamma_N)$ are denoted by $(\cdot\,,\cdot)$, $(\cdot\,,\cdot)_Q$, or $(\cdot\,,\cdot)_{\Gamma_N}$, respectively; the norm in $L^2(\Omega)$, $L^2(Q)$, or $L^2(\Gamma_N)$ is denoted by $\|\cdot\|$, $\|\cdot\|_Q$, or $\|\cdot\|_{\Gamma_N}$; the norm or seminorm in $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ is indicated by $\|\cdot\|_1$ or $|\cdot|_1$; the norm in $\boldsymbol{H}^{1/2}(\Gamma)$ or $H^{1/2}(\Gamma_D)$ is denoted by $\|\cdot\|_{1/2,\Gamma}$ or $\|\cdot\|_{1/2,\Gamma_D}$; the duality relation for the pair $X$ and $X^*$ is indicated by $\langle\cdot\,,\cdot\rangle_{X^*\times X}$ or $\langle\cdot\,,\cdot\rangle$. Let $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)\colon\ \operatorname{div}\boldsymbol{v} = 0\}$, $L_0^2(\Omega) = \{p \in L^2(\Omega)\colon\ (p, 1) = 0\}$, $\Theta = H^1(\Omega, \Gamma_D) \equiv \{S \in H^1(\Omega)\colon\ S|_{\Gamma_D} = 0\}$, $\tilde{\boldsymbol{H}}^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\colon\ (\boldsymbol{v}, \boldsymbol{n})_{\Gamma^{(i)}} = 0,\ i = 1, 2, \ldots, N,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_N} = 0\}$, and $\tilde{\boldsymbol{H}}^{1/2}(\Gamma) = \{\boldsymbol{g} = \boldsymbol{v}|_\Gamma\colon\ \boldsymbol{v} \in \tilde{\boldsymbol{H}}^1(\Omega)\}$. We also assume the following conditions to be satisfied:

1) $\Omega$ is a bounded domain in the space $\mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary $\Gamma \in C^{0,1}$ consisting of $N$ coupled components $\Gamma^{(i)}$ $(i = 1, 2, \ldots, N)$; $\Gamma_D \in C^{0,1}$, $\operatorname{meas}\Gamma_D > 0$, $\Gamma_N \in C^{0,1}$, $\Gamma_D \cap \Gamma_N = \varnothing$, and $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$;

2) $\boldsymbol{g} \in \tilde{\boldsymbol{H}}^{1/2}(\Gamma)$, $\beta_T \in L^2(\Omega)$, and $\psi \in H^{1/2}(\Gamma_D)$;

3) $\chi \in L^2(\Gamma_N)$;

4) $K \subset L^2(\Gamma_N)$ is a non-empty convex closed set.

The following lemma is valid (see [9]).

**Lemma 1.** *If condition* 1 *is satisfied, there exist* $\Omega$*-dependent constants* $\delta_0$, $\delta_1$, $\gamma_0$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, $\beta_1$, *and* $\beta$, *such that*

$$(\nabla\boldsymbol{v}, \nabla\boldsymbol{v}) \geqslant \delta_0\|\boldsymbol{v}\|_1^2 \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \qquad (\nabla S, \nabla S) \geqslant \delta_1\|S\|_1^2 \quad \forall\ S \in \Theta; \tag{1.4}$$

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w})| \leqslant \gamma_0\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \qquad |(\boldsymbol{u}\cdot\nabla T, S)| \le \gamma_1\|\boldsymbol{u}\|_1\|T\|_1\|S\|_1; \tag{1.5}$$

$$|(\beta_T\boldsymbol{G}T, \boldsymbol{v})| \le \beta_1\|\boldsymbol{v}\|_1\|T\|_1, \qquad |(\chi, S)_{\Gamma_N}| \leqslant \gamma_2\|\chi\|_{\Gamma_N}\|S\|_1; \tag{1.6}$$

$$\|\operatorname{rot}\boldsymbol{v}\| \le \gamma_3\|\boldsymbol{v}\|_1, \qquad \|\boldsymbol{v}\|_Q \leqslant \gamma_4\|\boldsymbol{v}\|_1, \qquad |(\zeta_d, \boldsymbol{v})_Q| \le \gamma_4\|\zeta_d\|_Q\|\boldsymbol{v}\|_1. \tag{1.7}$$

*The following relations are valid:*

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \quad \text{with} \quad \operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega); \tag{1.8}$$

$$(\boldsymbol{u}\cdot\nabla T, T) = 0 \quad \forall\ \boldsymbol{u} \in \tilde{\boldsymbol{H}}^1(\Omega) \quad \text{with} \quad \operatorname{div}\boldsymbol{u} = 0, \quad T \in \Theta; \tag{1.9}$$

$$\inf_{q \in L_0^2(\Omega),\ q \neq 0}\ \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v} \neq 0} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1\|q\|} \ge \beta = \text{const} > 0. \tag{1.10}$$

Multiplying the equations in system (1.1), (1.2) by the test functions and integrating them, we obtain a weak formulation for problem (1.1)–(1.3). In such a formulation, the solution of problem (1.1)–(1.3) is finding a triple $(\boldsymbol{u}, p, T) \in \tilde{\boldsymbol{H}}^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ satisfying the relations

$$\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - (\operatorname{div}\boldsymbol{v}, p) = ((1 - \beta_T T)\boldsymbol{G}, \boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega); \tag{1.11}$$

$$\lambda(\nabla T, \nabla S) + (\boldsymbol{u}\cdot\nabla T, S) = (\chi, S)_{\Gamma_N} \qquad \forall\ S \in \Theta; \tag{1.12}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma, \qquad T\Big|_{\Gamma_D} = \psi. \tag{1.13}$$

It is known (see, e.g., [9]) that there exists at least one solution $(\boldsymbol{u}, p, T)$ of problem (1.11)–(1.13) if conditions 1–3 are satisfied, and the following estimates are valid: $\|\boldsymbol{u}\|_1 \le M_{\boldsymbol{u}}$, $\|p\| \le M_p$, and $\|T\|_1 \le M_T$ ($M_{\boldsymbol{u}}$, $M_p$, and $M_T$

are non-decreasing continuous functions of the norms $\|\boldsymbol{g}\|_{1/2,\Gamma}$, $\|\beta_T\|$, $\|\psi\|_{1/2,\Gamma_D}$, and $\|\chi\|_{\Gamma_N}$). If the functions $\boldsymbol{g}$, $\psi$, and $\chi$ are "small" (or the viscosity $\nu$ is "large"), i.e., $\gamma_0 M_{\boldsymbol{u}} + (\beta_1\gamma_1/\delta_1\lambda)M_T < \delta_0\nu$ [the constants $\delta_i$, $\gamma_i$, and $\beta_1$ are introduced in Eqs. (1.4)–(1.6)], then the solution is unique.

**2. Formulation of the Extremum Problem. Properties of the Solution.** Let us divide the set of the initial data of problem (1.1)–(1.3) into two groups: group of controls, where we place the function $\chi$ playing the role of the sought control, and group of fixed data, where we place the non-changing functions $\beta_T$, $\boldsymbol{g}$, and $\psi$. Let $\boldsymbol{x} = (\boldsymbol{u}, p, T)$, $u_0 = (\beta_T, \boldsymbol{g}, \psi)$, and $u = \chi$. We also assume that the control $u$ can change on the set $K$ introduced in condition 4.

Assuming that $X = \tilde{\boldsymbol{H}}^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ and $Y = \boldsymbol{H}^{-1}(\Omega) \times L_0^2(\Omega) \times \tilde{\boldsymbol{H}}^{1/2}(\Gamma) \times \Theta^* \times H^{1/2}(\Gamma_D)$, we introduce the operator $F \equiv (F_1, F_2, F_3, F_4, F_5)$: $X \times K \to Y$, where

$$\langle F_1(\boldsymbol{x}, u), \boldsymbol{v}\rangle = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - (\operatorname{div}\boldsymbol{v}, p) - ((1-\beta_T T)\boldsymbol{G}, \boldsymbol{v}),$$

$$\langle F_4(\boldsymbol{x}, u), S\rangle = \lambda(\nabla T, \nabla S) + (\boldsymbol{u}\cdot\nabla T, S) - (\chi, S)_{\Gamma_N},$$

$$\langle F_2(\boldsymbol{x}, u), q\rangle = (\operatorname{div}\boldsymbol{u}, q), \qquad F_3(\boldsymbol{x}, u) = \boldsymbol{u}\Big|_\Gamma - \boldsymbol{g}, \qquad F_5(\boldsymbol{x}, u) = T\Big|_{\Gamma_D} - \psi.$$

Replacing Eqs. (1.11)–(1.13) by the operator equation $F(\boldsymbol{x}, u) \equiv F(\boldsymbol{u}, p, T, \chi) = 0$, we consider the extremum problem

$$J(\boldsymbol{x}, u) \equiv \frac{\mu_0}{2}\, I(\boldsymbol{u}) + \frac{\mu_1}{2}\,\|\chi\|^2_{\Gamma_N} \to \inf, \qquad F(\boldsymbol{x}, u) = 0, \qquad (\boldsymbol{x}, u) \in X \times K. \tag{2.1}$$

Here, $I(\boldsymbol{u})$ is the weakly lower semi-continuous cost functional on $\boldsymbol{H}^1(\Omega)$, $\mu_0$ and $\mu_1$ are positive dimensional parameters (see more details in [6, 7]). We use two cost functionals: $I_1(\boldsymbol{u}) = \|\boldsymbol{u} - \boldsymbol{v}_d\|^2_Q = \|\boldsymbol{u} - \boldsymbol{v}_d\|^2_{\boldsymbol{L}^2(Q)}$ and $I_2(\boldsymbol{u}) = \|\operatorname{rot}\boldsymbol{u} - \zeta_d\|^2_{\boldsymbol{L}^2(Q)}$, where the function $\boldsymbol{v}_d \in \boldsymbol{L}^2(Q)$ (or $\zeta_d \in \boldsymbol{L}^2(Q)$) models the velocity (or vorticity) field, which is measured or desired in a certain subdomain $Q$ of the domain $\Omega$.

Let us use $(\boldsymbol{x}_1, u_1) \equiv (\boldsymbol{u}_1, p_1, T_1, \chi_1) \in X \times K$ to indicate the (arbitrary) solution of the extremum problem (2.1); the existence of this solution follows from [9] if conditions 1, 2, and 4 are satisfied. We also use $(\boldsymbol{x}_2, u_2) \equiv (\boldsymbol{u}_2, p_2, T_2, \chi_2) \in X \times K$ to indicate the solution of the extremum problem close to (2.1)

$$\tilde{J}(\boldsymbol{x}, u) = \frac{\mu_0}{2}\, \tilde{I}(\boldsymbol{u}) + \frac{\mu_1}{2}\,\|\chi\|^2_{\Gamma_N} \to \inf, \qquad F(\boldsymbol{x}, u) = 0, \qquad (\boldsymbol{x}, u) \in X \times K, \tag{2.2}$$

obtained by replacing the functional $I$ in Eq. (2.1) by a similar functional $\tilde{I}$. We derive an important inequality satisfied by the difference in the solutions $(\boldsymbol{x}_1, u_1)$ and $(\boldsymbol{x}_2, u_2)$ of problems (2.1) and (2.2).

By virtue of results described in Sec. 1, the following estimates are valid for the triples $(\boldsymbol{u}_i, p_i, T_i)$:

$$\|\boldsymbol{u}_i\|_1 \le M^0_{\boldsymbol{u}}, \qquad \|p_i\| \le M^0_p, \qquad \|T_i\|_1 \le M^0_T \tag{2.3}$$

$[M^0_{\boldsymbol{u}} = \sup_{u\in K} M_{\boldsymbol{u}}(u_0, u)$, $M^0_p = \sup_{u\in K} M_p(u_0, u)$, and $M^0_T = \sup_{u\in K} M_T(u_0, u)]$. We introduce the model Reynolds, Rayleigh, and Prandtl numbers

$$\mathrm{Re} = \frac{\gamma_0 M^0_{\boldsymbol{u}}}{\delta_0\nu}, \qquad \mathrm{R} = \frac{\gamma_1}{\delta_0\nu}\,\frac{\beta_1 M^0_T}{\delta_1\lambda}, \qquad \mathrm{Pr} = \frac{\delta_0\nu}{\delta_1\lambda} \tag{2.4}$$

and assume the following condition to be valid:

$$\mathrm{Re} + \mathrm{R} \equiv \frac{\gamma_0 M^0_{\boldsymbol{u}}}{\delta_0\nu} + \frac{\gamma_1}{\delta_0\nu}\,\frac{\beta_1 M^0_T}{\delta_1\lambda} < \frac{1}{2}. \tag{2.5}$$

Let us use $(1, \boldsymbol{y}_i^*)$ $(\boldsymbol{y}_i^* \equiv (\xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^t) \in \boldsymbol{V} \times L_0^2(\Omega) \times \tilde{\boldsymbol{H}}^{1/2}(\Gamma)^* \times \Theta \times H^{1/2}(\Gamma_D)^*$, $i = 1, 2)$ to indicate nontrivial Lagrangian multipliers corresponding to the solutions $(\boldsymbol{x}_i, u_i)$ and satisfying the relations [9]

$$\nu(\nabla \boldsymbol{w}, \nabla\xi_i) + ((\boldsymbol{u}_i\cdot\nabla)\boldsymbol{w}, \xi_i) + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_i, \xi_i) + \varkappa(\boldsymbol{w}\cdot\nabla T_i, \theta_i) - (\operatorname{div}\boldsymbol{w}, \sigma_i) + \langle\zeta_i, \boldsymbol{w}\rangle_\Gamma$$

$$= -(\mu_0/2)\langle (I^i)'_{\boldsymbol{u}}(\boldsymbol{x}_i, u_i), \boldsymbol{w}\rangle \qquad \forall\ \boldsymbol{w} \in \tilde{\boldsymbol{H}}^1(\Omega), \quad i = 1, 2; \tag{2.6}$$

$$\varkappa[\lambda(\nabla\tau, \nabla\theta_i) + (\boldsymbol{u}_i\cdot\nabla\tau, \theta_i) + \langle\zeta_i^t, \tau\rangle_{\Gamma_D}] + (\beta_T\boldsymbol{G}\tau, \xi_i) = 0 \qquad \forall\ \tau \in H^1(\Omega); \tag{2.7}$$

$$(\mu_1\chi_i - \varkappa\theta_i, \tilde{\chi} - \chi_i)_{\Gamma_N} \ge 0 \qquad \forall\ \tilde{\chi} \in K. \tag{2.8}$$

Here, $\varkappa$ is an auxiliary dimensional multiplier; the functions $\xi_i$ and $\theta_i$ are considered as adjoint velocity and temperature. In Eqs. (2.6), we have $I^1 \equiv I$, $I^2 \equiv \tilde{I}$. Let us assume that $\chi = \chi_1 - \chi_2$, $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $p = p_1 - p_2$, $T = T_1 - T_2$, $\xi = \xi_1 - \xi_2$, $\sigma = \sigma_1 - \sigma_2$, $\zeta = \zeta_1 - \zeta_2$, $\theta = \theta_1 - \theta_2$, $\zeta^t = \zeta_1^t - \zeta_2^t$ and subtract relations (1.11)–(1.13) written for $\boldsymbol{u}_2$, $p_2$, $T_2$, and $u_2$ from relations (1.11)–(1.13) written for $\boldsymbol{u}_1$, $p_1$, $T_1$, and $u_1$. As a result, we obtain

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_1 + (\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, p) + (\beta_T \boldsymbol{G} T, \boldsymbol{v}) = 0 \qquad \forall\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega); \tag{2.9}$$

$$\lambda(\nabla T, \nabla S) + (\boldsymbol{u} \cdot \nabla T_1, S) + (\boldsymbol{u}_2 \cdot \nabla T, S) = (\chi, S)_{\Gamma_N} \qquad \forall\ S \in \Theta; \tag{2.10}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{u}\Big|_{\Gamma} = \boldsymbol{0}, \qquad T\Big|_{\Gamma_D} = 0. \tag{2.11}$$

Assuming that $\tilde{\chi} = \chi_1$ at $i = 2$ and $\tilde{\chi} = \chi_2$ at $i = 1$ in inequality (2.8) and adding these relations, we obtain

$$-\varkappa(\chi, \theta)_{\Gamma_N} \le -\mu_1 \|\chi\|_{\Gamma_N}^2. \tag{2.12}$$

We subtract identities (2.6) and (2.7) written for $(\boldsymbol{x}_1, u_1, \boldsymbol{y}_1^*)$ and $(\boldsymbol{x}_2, u_2, \boldsymbol{y}_2^*)$ one from the other. After that, assuming that $\boldsymbol{w} = \boldsymbol{u}$ and $\tau = T$, we add these identities. Taking into account Eq. (2.12), we obtain

$$\nu(\nabla \boldsymbol{u}, \nabla \xi) + ((\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_1, \xi) + 2((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \xi_2) + \varkappa(\boldsymbol{u} \cdot \nabla T_1, \theta) + \varkappa(\boldsymbol{u} \cdot \nabla T, \theta_2)$$
$$+ \varkappa[\lambda(\nabla T, \nabla \theta) + (\boldsymbol{u}_1 \cdot \nabla T, \theta) + (\boldsymbol{u} \cdot \nabla T, \theta_2)] + (\beta_T \boldsymbol{G} T, \xi)$$
$$= -(\mu_0/2)\langle I'_{\boldsymbol{u}}(\boldsymbol{u}_1) - \tilde{I}'_{\boldsymbol{u}}(\boldsymbol{u}_2), \boldsymbol{u}\rangle. \tag{2.13}$$

We assume that $\boldsymbol{v} = \xi$ in Eq. (2.9) and $S = \varkappa\theta$ in Eq. (2.10) and subtract the resultant relations from Eq. (2.13). Using inequality (2.12) and following [7], we find

$$((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \xi_1 + \xi_2) + \varkappa(\boldsymbol{u} \cdot \nabla T, \theta_1 + \theta_2) + (\mu_0/2)\langle I'_{\boldsymbol{u}}(\boldsymbol{u}_1) - \tilde{I}'_{\boldsymbol{u}}(\boldsymbol{u}_2), \boldsymbol{u}\rangle \le -\mu_1 \|\chi\|_{\Gamma_N}^2. \tag{2.14}$$

The result obtained is formulated as a theorem.

**Theorem 1.** *Let the quadruples* $(\boldsymbol{u}_1, p_1, T_1, \chi_1)$ *and* $(\boldsymbol{u}_2, p_2, T_2, \chi_2)$ *with conditions* 1, 2, 4, *and* (2.5) *being satisfied, be solutions of problems* (2.1) *and* (2.2), $\boldsymbol{y}_i^* = (\xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^t)$ $(i = 1, 2)$ *being the corresponding Lagrangian multipliers. Then, relation* (2.14) *is valid for the differences* $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $p = p_1 - p_2$, $T = T_1 - T_2$, *and* $\chi = \chi_1 - \chi_2$.

Assuming that $\boldsymbol{v} = \boldsymbol{u}$ in Eq. (2.9) and taking into account Eq. (1.8), we find $\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) = -((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_1, \boldsymbol{u}) - (\beta_T \boldsymbol{G} T, \boldsymbol{u})$. From here, using Eqs. (1.4)–(1.6), we obtain

$$\delta_0 \nu \|\boldsymbol{u}\|_1^2 \le \gamma_0 M_{\boldsymbol{u}}^0 \|\boldsymbol{u}\|_1^2 + \beta_1 \|T\|_1 \|\boldsymbol{u}\|_1. \tag{2.15}$$

It follows from Eq. (2.5) that

$$\frac{\delta_0 \nu}{2} < \delta_0 \nu - \gamma_0 M_{\boldsymbol{u}}^0 - \frac{\beta_1 \gamma_1}{\delta_1 \lambda}\, M_T^0 \leqslant \delta_0 \nu - \gamma_0 M_{\boldsymbol{u}}^0. \tag{2.16}$$

Taking into account Eq. (2.16), we write Eq. (2.15) in the form $(\delta_0\nu/2)\|\boldsymbol{u}\|_1^2 \le (\delta_0\nu - \gamma_0 M_{\boldsymbol{u}}^0)\|\boldsymbol{u}\|_1^2 \le \beta_1 \|T\|_1 \|\boldsymbol{u}\|_1$. As a result, we obtain the following estimate for the norm of the difference $\boldsymbol{u}$:

$$\|\boldsymbol{u}\|_1 \le (2\beta_1/\delta_0\nu)\|T\|_1. \tag{2.17}$$

A similar estimate is also valid for the pressure difference $p = p_1 - p_2$. To find it, we use the inf-sup-condition (1.10), which reads that, for the function $p = p_1 - p_2$ and any (arbitrarily small) number $\delta > 0$, there exists a function $\boldsymbol{v}_0 \in \boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{v}_0 \ne 0$, such that $(\operatorname{div} \boldsymbol{v}_0, p) \ge \beta_0 \|\boldsymbol{v}_0\|_1 \|p\|$ and $\beta_0 = \beta - \delta > 0$. Assuming that $\boldsymbol{v} = \boldsymbol{v}_0$ in the identity for $\boldsymbol{u}$ in Eq. (2.9) and using this estimate, as well as estimates (1.4) and (1.5), we obtain

$$\beta_0 \|\boldsymbol{v}_0\|_1 \|p\| \le (\operatorname{div} \boldsymbol{v}_0, p) \le (\nu + 2\gamma_0 M_{\boldsymbol{u}}^0)\|\boldsymbol{v}_0\|_1 \|\boldsymbol{u}\|_1 + \beta_1 \|T\|_1 \|\boldsymbol{v}_0\|_1. \tag{2.18}$$

Eliminating $\|\boldsymbol{v}_0\|_1 \ne 0$ in Eq. (2.18), we have

$$\|p\| \le \frac{\nu + 2\gamma_0 M_{\boldsymbol{u}}^0}{\beta_0}\, \|\boldsymbol{u}\|_1 + \frac{\beta_1}{\beta_0}\, \|T\|_1 \le \frac{\beta_1}{\beta_0}\, (2M + 1)\|T\|_1, \qquad M = \delta_0^{-1} + 2\,\mathrm{Re}\,. \tag{2.19}$$

**3. Uniqueness and Stability of Solutions of Extremum Problems.** Using Theorem 1 and estimates (2.17) and (2.19), we study the uniqueness and stability of the solution of problem (2.1) for particular cost functionals. Let us first consider the extremum problem

$$J(\boldsymbol{v},\chi) \equiv \frac{\mu_0}{2}\,\|\boldsymbol{v}-\boldsymbol{v}_d\|_Q^2 + \frac{\mu_1}{2}\,\|\chi\|_{\Gamma_N}^2 \to \inf, \qquad F(\boldsymbol{x},\chi)=0, \qquad \boldsymbol{x}\in X, \quad \chi\in K, \tag{3.1}$$

corresponding to the functional $I_1$. Let $(\boldsymbol{x}_1,u_1)\equiv(\boldsymbol{u}_1,p_1,T_1,\chi_1)$ be the solution of problem (3.1) corresponding to the function $\boldsymbol{v}_d\equiv\boldsymbol{u}_d^{(1)}\in\boldsymbol{L}^2(Q)$ and $(\boldsymbol{x}_2,u_2)\equiv(\boldsymbol{u}_2,p_2,T_2,\chi_2)$ be the solution of problem (3.1) corresponding to the disturbed function $\tilde{\boldsymbol{v}}_d\equiv\boldsymbol{u}_d^{(2)}\in\boldsymbol{L}^2(Q)$. Assuming that $\boldsymbol{u}_d=\boldsymbol{u}_d^{(1)}-\boldsymbol{u}_d^{(2)}$, we obtain the following identities for problem (3.1):

$$\langle I'_{\boldsymbol{u}}(\boldsymbol{u}_i),\boldsymbol{w}\rangle = 2(\boldsymbol{u}_i-\boldsymbol{u}_d^{(i)},\boldsymbol{w})_Q, \qquad \langle I'_{\boldsymbol{u}}(\boldsymbol{u}_1)-\tilde{I}'_{\boldsymbol{u}}(\boldsymbol{u}_2),\boldsymbol{u}\rangle = 2(\|\boldsymbol{u}\|_Q^2-(\boldsymbol{u},\boldsymbol{u}_d)_Q). \tag{3.2}$$

Relations (2.7), (2.9)–(2.11) for problem (3.1) are not changed, while identity (2.6) and inequality (2.14) take the following form in view of Eq. (3.2):

$$\nu(\nabla\boldsymbol{w},\nabla\xi_i)+((\boldsymbol{u}_i\cdot\nabla)\boldsymbol{w},\xi_i)+((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_i,\xi_i)+\varkappa(\boldsymbol{w}\cdot\nabla T_i,\theta_i)-(\operatorname{div}\boldsymbol{w},\sigma_i)+\langle\zeta_i,\boldsymbol{w}\rangle_\Gamma$$

$$= -\mu_0(\boldsymbol{u}_i-\boldsymbol{u}_d^{(i)},\boldsymbol{w})_Q \qquad \forall\ \boldsymbol{w}\in\tilde{\boldsymbol{H}}^1(\Omega); \tag{3.3}$$

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\xi_1+\xi_2)+\varkappa(\boldsymbol{u}\cdot\nabla T,\theta_1+\theta_2)+\mu_0(\|\boldsymbol{u}\|_Q^2-(\boldsymbol{u},\boldsymbol{u}_d)_Q)\le-\mu_1\|\chi\|_{\Gamma_N}^2. \tag{3.4}$$

By virtue of Eq. (2.11), we have $T\in\Theta$. Assuming that $S=T$ in Eq. (2.10), we obtain the following relation by virtue of Eq. (1.9):

$$\lambda(\nabla T,\nabla T) = -(\boldsymbol{u}\cdot\nabla T_1,T)+(\chi,T)_{\Gamma_N}. \tag{3.5}$$

Using relations (1.4)–(1.6) and (2.3), we obtain the following relation from Eq. (3.5):

$$\delta_1\lambda\|T\|^2 \le \gamma_1 M_T^0\|\boldsymbol{u}\|_1\|T\|_1+\gamma_2\|\chi\|_{\Gamma_N}\|T\|_1.$$

From here and also from Eq. (2.17), it follows that

$$\|T\|_1 \le \frac{\gamma_1 M_T^0}{\delta_1\lambda}\,\|\boldsymbol{u}\|_1+\frac{\gamma_2}{\delta_1\lambda}\,\|\chi\|_{\Gamma_N} \leqslant \frac{2\beta_1}{\delta_0\nu}\,\frac{\gamma_1 M_T^0}{\delta_1\lambda}\,\|T\|_1+\frac{\gamma_2}{\delta_1\lambda}\,\|\chi\|_{\Gamma_N}. \tag{3.6}$$

Taking into account relations (2.4), (2.17), and (2.19), we obtain the following estimates from Eq. (3.6):

$$\|T\|_1 \le \frac{\gamma_2\|\chi\|_{\Gamma_N}}{\delta_1\lambda(1-2\,\mathrm{R})}, \qquad \|\boldsymbol{u}\|_1 \le \frac{2\beta_1\gamma_2\|\chi\|_{\Gamma_N}}{\delta_0\nu\delta_1\lambda(1-2\,\mathrm{R})}, \qquad \|p\| \leqslant \frac{\beta_1\gamma_2(2M+1)\|\chi\|_{\Gamma_N}}{\beta_0\delta_1\lambda(1-2\,\mathrm{R})}. \tag{3.7}$$

Assuming that $\boldsymbol{w}=\xi_i$ and $\tau=\theta_i$ in Eqs. (3.3) and (2.7) and taking into account Eqs. (1.8) and (1.9), we obtain

$$\nu(\nabla\xi_i,\nabla\xi_i) = -((\xi_i\cdot\nabla)\boldsymbol{u}_i,\xi_i)-\varkappa(\xi_i\cdot\nabla T_i,\theta_i)-\mu_0(\boldsymbol{u}_i-\boldsymbol{u}_d^{(i)},\xi_i)_Q; \tag{3.8}$$

$$\varkappa\lambda(\nabla\theta_i,\nabla\theta_i) = -(\beta_T\boldsymbol{G}\theta_i,\xi_i), \qquad i=1,2. \tag{3.9}$$

Using estimates (1.4)–(1.7), we have

$$(\nabla\xi_i,\nabla\xi_i)\ge\delta_0\|\xi_i\|_1^2, \qquad |((\xi_i\cdot\nabla)\boldsymbol{u}_i,\xi_i)|\le\gamma_0\|\boldsymbol{u}_i\|_1\|\xi_i\|_1^2\le\gamma_0 M_{\boldsymbol{u}}^0\|\xi_i\|_1^2; \tag{3.10}$$

$$|(\beta_T\boldsymbol{G}\theta_i,\xi_i)|\le\beta_1\|\theta_i\|_1\|\xi_i\|_1, \qquad \varkappa|(\xi_i\cdot\nabla T_i,\theta_i)|\le\varkappa\gamma_1 M_T^0\|\xi_i\|_1\|\theta_i\|_1; \tag{3.11}$$

$$|(\boldsymbol{u}_i-\boldsymbol{u}_d^{(i)},\xi_i)_Q|\le\|\boldsymbol{u}_i-\boldsymbol{u}_d^{(i)}\|_Q\|\xi_i\|_Q\le\gamma_4(\gamma_4 M_{\boldsymbol{u}}^0+\|\boldsymbol{u}_d^{(i)}\|_Q)\|\xi_i\|_1. \tag{3.12}$$

Taking into account Eqs. (3.10)–(3.12), we obtain the following expressions from Eqs. (3.8) and (3.9):

$$\begin{gathered}\|\theta_i\|_1\le\frac{\beta_1}{\delta_1\lambda\varkappa}\,\|\xi_i\|_1,\\ \Big(\delta_0\nu-\gamma_0 M_{\boldsymbol{u}}^0-\frac{\beta_1\gamma_1}{\delta_1\lambda}\,M_T^0\Big)\|\xi_i\|_1^2\le\mu_0\gamma_4(\gamma_4 M_{\boldsymbol{u}}^0+\|\boldsymbol{u}_d^{(i)}\|_Q)\|\xi_i\|_1.\end{gathered} \tag{3.13}$$

Using Eqs. (2.16) and (1.7), we subsequently derive the following relations from inequalities (3.13):

$$\|\xi_i\|_1 \le \frac{2\mu_0\gamma_4}{\delta_0\nu}\,(\gamma_4 M_{\boldsymbol{u}}^0 + \|\boldsymbol{u}_d^{(i)}\|_Q) = \frac{2\mu_0\gamma}{\gamma_0}\,(\mathrm{Re} + \mathrm{Re}^0),$$
$$\|\theta_i\|_1 \le \frac{2\mu_0\gamma\beta_1}{\gamma_0\delta_1\lambda\varkappa}\,(\mathrm{Re} + \mathrm{Re}^0), \tag{3.14}$$

where

$$\gamma = \gamma_4^2, \qquad \mathrm{Re}^0 = \frac{\gamma_0}{\delta_0\nu\gamma_4}\max\,(\|\boldsymbol{u}_d^{(1)}\|_Q, \|\boldsymbol{u}_d^{(2)}\|_Q). \tag{3.15}$$

Taking into account Eqs. (1.5) and (3.7) and also the inequalities for $\xi_i$ and $\theta_i$ in Eq. (3.14), we obtain

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \xi_1+\xi_2)| \le \gamma_0\|\boldsymbol{u}\|_1^2(\|\xi_1\|_1 + \|\xi_2\|_1) \le 4\mu_0\gamma\Big(\frac{2\beta_1}{\delta_0\nu}\,\frac{\gamma_2}{\delta_1\lambda}\Big)^2\frac{\mathrm{Re} + \mathrm{Re}^0}{(1-2\,\mathrm{R})^2}\,\|\chi\|_{\Gamma_N}^2,$$
$$\varkappa|(\boldsymbol{u}\cdot\nabla T, \theta_1+\theta_2)| \le 4\mu_0\gamma\,\frac{2\beta_1}{\delta_0\nu}\Big(\frac{\gamma_2}{\delta_1\lambda}\Big)^2\frac{\gamma_1\beta_1}{\delta_1\lambda}\,\frac{\mathrm{Re} + \mathrm{Re}^0}{\gamma_0(1-2\,\mathrm{R})^2}\,\|\chi\|_{\Gamma_N}^2 \tag{3.16}$$

[$\gamma$ and $\mathrm{Re}^0$ were defined in Eq. (3.15)]. From Eqs. (3.16) and (2.4), we obtain

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \xi_1+\xi_2) + \varkappa(\boldsymbol{u}\cdot\nabla T, \theta_1+\theta_2)| \leqslant 2\mu_0\gamma\Big(\frac{2\beta_1}{\delta_0\nu}\,\frac{\gamma_2}{\delta_1\lambda}\Big)^2\frac{\mathrm{Re} + \mathrm{Re}^0}{(1-2\,\mathrm{R})^2}\Big(2 + \frac{\gamma_1}{\mathrm{Pr}\,\gamma_0}\Big)\|\chi\|_{\Gamma_N}^2. \tag{3.17}$$

Let the initial data for problem (3.1) and the parameters $\mu_0$ and $\mu_1$ be such that

$$(1-\varepsilon)\mu_1 \ge 2\mu_0\gamma\Big(\frac{2\beta_1}{\delta_0\nu}\,\frac{\gamma_2}{\delta_1\lambda}\Big)^2\frac{\mathrm{Re} + \mathrm{Re}^0}{(1-2\,\mathrm{R})^2}\Big(2 + \frac{\gamma_1}{\mathrm{Pr}\,\gamma_0}\Big), \qquad \varepsilon = \mathrm{const} > 0. \tag{3.18}$$

If condition (3.18) is satisfied, then it follows from inequality (3.17) that

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \xi_1+\xi_2) + \varkappa(\boldsymbol{u}\cdot\nabla T, \theta_1+\theta_2)| \le (1-\varepsilon)\mu_1\|\chi\|_{\Gamma_N}^2. \tag{3.19}$$

In view of Eq. (3.19), we obtain the following inequality from Eq. (3.4):

$$\mu_0(\|\boldsymbol{u}\|_Q^2 - (\boldsymbol{u}, u_d)_Q) \leqslant -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \xi_1+\xi_2) - \varkappa(\boldsymbol{u}\cdot\nabla T, \theta_1+\theta_2) - \mu_1\|\chi\|_{\Gamma_N} \leqslant -\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2. \tag{3.20}$$

It follows from this inequality that $\|\boldsymbol{u}\|_Q^2 \le (\boldsymbol{u}, \boldsymbol{u}_d)_Q \le \|\boldsymbol{u}\|_Q\|\boldsymbol{u}_d\|_Q$. From here, we obtain $\|\boldsymbol{u}\|_Q \le \|\boldsymbol{u}_d\|_Q$. Then, by virtue of $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\boldsymbol{u}_d = \boldsymbol{u}_d^{(1)} - \boldsymbol{u}_d^{(2)}$, we have

$$\|\boldsymbol{u}\|_Q \equiv \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_Q \le \|\boldsymbol{u}_d^{(1)} - \boldsymbol{u}_d^{(2)}\|_Q \equiv \|\boldsymbol{u}_d\|_Q. \tag{3.21}$$

At $Q = \Omega$, estimate (3.21) is the estimate of stability of the component $\hat{\boldsymbol{u}}$ of the solution of problem (3.1) with respect to small perturbations in the norm $\boldsymbol{L}^2(\Omega)$ of the function $\boldsymbol{v}_d \in \boldsymbol{L}^2(\Omega)$. At $\boldsymbol{u}_d^{(1)} = \boldsymbol{u}_d^{(2)}$, it follows from Eq. (3.21) that $\boldsymbol{u}_1 = \boldsymbol{u}_2$. In turn, from here, in view of Eqs. (2.17), (2.19), and (3.20), it follows that $T_1 = T_2$, $p_1 = p_2$ and $\chi_1 = \chi_2$, which means that the solution of problem (3.1) is unique if $Q = \Omega$ and condition (3.18) is satisfied.

Note that the solution of problem (3.1) under condition (3.18) is also unique and stable if $Q \subset \Omega$, i.e., if $Q$ is only part of the domain $\Omega$. To prove it, we use inequality (3.20), which can be written in the following form with allowance for Eq. (3.21):

$$\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2 \le -\mu_0\|\boldsymbol{u}\|_Q^2 + \mu_0\|\boldsymbol{u}\|_Q\|\boldsymbol{u}_d\|_Q \le \mu_0\|\boldsymbol{u}_d\|_Q^2. \tag{3.22}$$

Using Eqs. (3.22) and (3.7), we obtain the following stability estimates:

$$\|\chi_1 - \chi_2\|_{\Gamma_N} \le \sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\,\Delta, \qquad \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1 \le \frac{2\beta_1\gamma_2\Delta}{\delta_0\nu\delta_1\lambda(1-2\,\mathrm{R})}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}},$$
$$\|T_1 - T_2\|_1 \le \frac{\gamma_2\Delta}{\delta_1\lambda(1-2\,\mathrm{R})}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}, \qquad \|p_1 - p_2\| \le \frac{\beta_1\gamma_2(2M+1)}{\beta_0\delta_1\lambda(1-2\,\mathrm{R})}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\,\Delta \tag{3.23}$$

($\Delta = \|\boldsymbol{u}_d^{(1)} - \boldsymbol{u}_d^{(2)}\|_Q$). Thus, the following theorem is proved.

**Theorem 2.** *Let the quadruple* $(\boldsymbol{u}_i, p_i, T_i, \chi_i)$, *with conditions* 1, 2, 4, *and* (2.5) *being satisfied, be a solution of problem* (3.1) *corresponding to a given function* $\boldsymbol{u}_d^{(i)} \in \boldsymbol{L}^2(Q)$ $(i = 1, 2)$, *and let condition* (3.18) *be satisfied. Then, the stability estimates* (3.21) *and* (3.23), *where* $\Delta = \|\boldsymbol{u}_d^{(1)} - \boldsymbol{u}_d^{(2)}\|_Q$, *are valid.*

Using a similar scheme, we can study the extremum problem

$$J(\boldsymbol{v}, \chi) \equiv \frac{\mu_0}{2} \|\mathrm{rot}\, \boldsymbol{v} - \zeta_d\|_Q^2 + \frac{\mu_1}{2} \|\chi\|_{\Gamma_N}^2 \to \inf, \quad F(\boldsymbol{x}, \chi) = 0, \quad (\boldsymbol{x}, \chi) \in X \times K, \tag{3.24}$$

obtained from Eq. (3.1) by replacing the functional $I_1(\boldsymbol{v})$ with $I_2(\boldsymbol{v})$. A similar analysis shows that the following theorem is valid.

**Theorem 3.** *Let the quadruple* $(\boldsymbol{u}_i, p_i, T_i, \chi_i)$, *with conditions* 1, 2, 4, *and* (2.5) *being satisfied, be a solution of problem* (3.24) *corresponding to a given function* $\zeta_d^{(i)} \in \boldsymbol{L}^2(Q)$ $(i = 1, 2)$, *and let condition* (3.18), *where* $\gamma = \gamma_3^2$, $\mathrm{Re}^0 = (\gamma_0/(\delta_0 \nu \gamma_3)) \max(\|\zeta_d^{(1)}\|_Q, \|\zeta_d^{(2)}\|_Q)$, *be satisfied. Then, we have* $\|\mathrm{rot}\, \boldsymbol{u}_1 - \mathrm{rot}\, \boldsymbol{u}_2\|_Q \leqslant \|\zeta_d^{(1)} - \zeta_d^{(1)}\|_Q$, *and stability estimates* (3.23) *are valid for* $\Delta = \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q$.

Note that the uniqueness and stability of the solution of problem (3.1) or (3.24), both at $Q = \Omega$ and $Q \subset \Omega$, can be proved only under the condition that the parameter $\mu_1$ in Eq. (3.1) or (3.24) is positive and satisfies condition (3.18). Conditions (2.5) and (3.18) are sufficiently stiff, because they actually have the physical meaning of smallness conditions imposed on the initial data. For particular extremum problems, these conditions can be substantially attenuated, as it is demonstrated below in analyzing computational experiments.

**4. Numerical Algorithm. Analysis of Computational Experiments.** Let us discuss the results of computational experiments aimed at solving the extremum problem (3.24) with $K = L^2(\Gamma_N)$. In this case, inequality (2.8) transforms to the identity $(\mu_1 \chi - \theta, \tilde{\chi})_{\Gamma_N} = 0$ for all $\tilde{\chi} \in L^2(\Gamma_N)$, which implies that the control $\chi$ is related to the adjoint temperature $\theta$ by the formula $\chi = \theta/\mu_1$. Eliminating $\chi$, we write the remaining relations for the sextuple $(\boldsymbol{u}, p, T, \xi, \sigma, \theta)$ in the form of the operator equation $\Phi(\boldsymbol{u}, p, T, \xi, \sigma, \theta) = 0$ ($\Phi$ is a nonlinear operator defined by the operator $F$ introduced above). Taking into account the nonlinearity of the operator $\Phi$, we use an iterative algorithm based on Newton's method to solve the resultant equation numerically. This algorithm includes the following stages:

1. Choose the initial guess $\boldsymbol{u}_0$, $p_0$, $T_0$, $\xi_0$, $\sigma_0$, $\theta_0$. Assume that $n = 0$.

2. Calculate $\tilde{\boldsymbol{u}}$, $\tilde{p}$, $\tilde{T}$, $\tilde{\xi}$, $\tilde{\sigma}$, $\tilde{\theta}$ by solving the linear operator equation $\Phi'(\boldsymbol{u}_n, p_n, T_n, \xi_n, \sigma_n, \theta_n)(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{T}, \tilde{\xi}, \tilde{\sigma}, \tilde{\theta}) = -\Phi(\boldsymbol{u}_n, p_n, T_n, \xi_n, \sigma_n, \theta_n)$, in the weak formulation of the corresponding boundary-value problem with respect to the functions $\tilde{\boldsymbol{u}}$, $\tilde{p}$, $\tilde{T}$, $\tilde{\xi}$, $\tilde{\sigma}$, and $\tilde{\theta}$, which has the form

$$\begin{gathered}
\nu(\nabla \tilde{\boldsymbol{u}}, \nabla \boldsymbol{v}) + ((\tilde{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_n, \boldsymbol{v}) + ((\boldsymbol{u}_n \cdot \nabla)\tilde{\boldsymbol{u}}, \boldsymbol{v}) - (\tilde{p}, \mathrm{div}\, \boldsymbol{v}) + (\beta_T \tilde{T} \boldsymbol{G}, \boldsymbol{v}) \\
+ \nu(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n, \boldsymbol{v}) - (p_n, \mathrm{div}\, \boldsymbol{v}) - ((1 - \beta_T T_n)\boldsymbol{G}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\
\lambda(\nabla \tilde{T}, \nabla S) + (\tilde{\boldsymbol{u}} \cdot \nabla T_n,\, S) + (\boldsymbol{u}_n \cdot \nabla \tilde{T},\, S) - (1/\mu_1)(\tilde{\theta}, S)_{\Gamma_N} \\
+ \lambda(\nabla T_n, \nabla S) + (\boldsymbol{u}_n \cdot \nabla T_n,\, S) - (1/\mu_1)(\theta_n, S)_{\Gamma_N} = 0 \qquad \forall\, S \in H^1(\Omega, \Gamma_D), \\
\nu(\nabla \boldsymbol{w}, \nabla \tilde{\xi}) + ((\tilde{\boldsymbol{u}} \cdot \nabla)\boldsymbol{w}, \xi_n) + ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{w}, \tilde{\xi}) + ((\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}, \xi_n) + ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_n, \tilde{\xi}) \\
+ (\boldsymbol{w} \cdot \nabla \tilde{T}, \theta_n) + (\boldsymbol{w} \cdot \nabla T_n, \tilde{\theta}) + (\tilde{\sigma}, \mathrm{div}\, \boldsymbol{w}) + (\mu_0/2)(I_i)''_{\boldsymbol{u}}(\boldsymbol{u}_n)(\tilde{\boldsymbol{u}}, \boldsymbol{w}) \\
+ \nu(\nabla \boldsymbol{w}, \nabla \xi_n) + ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{w}, \xi_n) + ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_n, \xi_n) \\
+ (\boldsymbol{w} \cdot \nabla T_n, \theta_n) + (\sigma_n, \mathrm{div}\, \boldsymbol{w}) + (\mu_0/2)\langle (I_i)'_{\boldsymbol{u}}(\boldsymbol{u}_n), \boldsymbol{w} \rangle = 0 \qquad \forall\, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \\
\lambda(\nabla h, \nabla \tilde{\theta}) + (\tilde{\boldsymbol{u}} \cdot \nabla h, \theta_n) + (\boldsymbol{u}_n \cdot \nabla h, \tilde{\theta}) + (\beta h \boldsymbol{G}, \tilde{\xi}) \\
+ \lambda(\nabla h, \nabla \theta_n) + (\boldsymbol{u}_n \cdot \nabla h, \theta_n) + (\beta h \boldsymbol{G}, \xi_n) = 0 \qquad \forall\, h \in H^1(\Omega, \Gamma_D), \\
\mathrm{div}\, \tilde{\boldsymbol{u}} = -\mathrm{div}\, \boldsymbol{u}_n \quad \text{in} \quad \Omega, \qquad \tilde{\boldsymbol{u}} = \boldsymbol{g} - \boldsymbol{u}_n \quad \text{on} \quad \Gamma, \qquad \tilde{T} = \psi - T_n \quad \text{on} \quad \Gamma_D, \\
\mathrm{div}\, \tilde{\xi} = -\mathrm{div}\, \xi_n \quad \text{in} \quad \Omega, \qquad \tilde{\xi} = -\xi_n \quad \text{on} \quad \Gamma, \qquad \tilde{\theta} = -\theta_n \quad \text{on} \quad \Gamma_D.
\end{gathered} \tag{4.1}$$

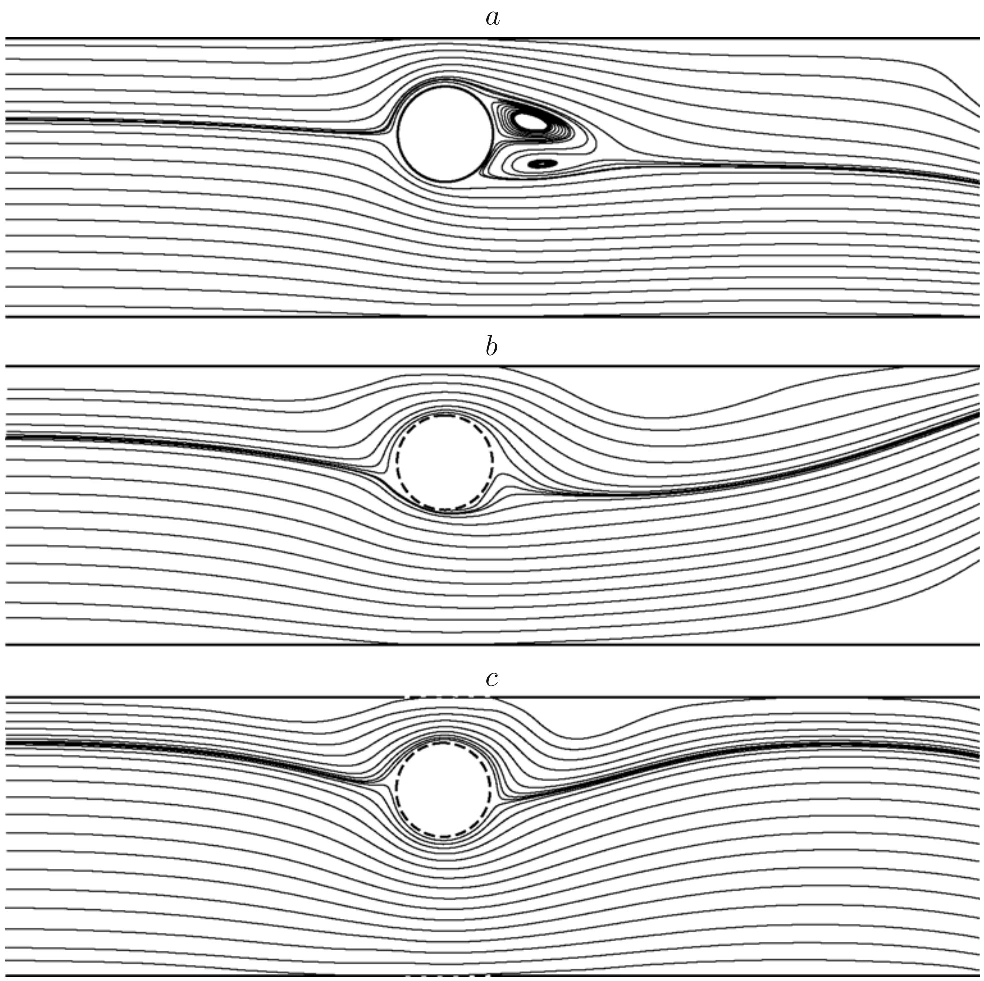


Fig. 1. Streamlines in the flow around a cylinder in a channel at Re = 100: (a) uncontrolled flow; (b, c) controlled flow at $\mathrm{R} = 10^5$ with control through the body surface (b) and through both the body surface and some part of the channel walls (c).

3. Recalculate the values of the sought quantities by the formulas $\boldsymbol{u}_{n+1}=\boldsymbol{u}_n+\tilde{\boldsymbol{u}}$, $p_{n+1} = p_n+\tilde{p}$, $T_{n+1} = T_n+\tilde{T}$, $\xi_{n+1} = \xi_n + \tilde{\xi}$, $\sigma_{n+1} = \sigma_n + \tilde{\sigma}$, and $\theta_{n+1} = \theta_n + \tilde{\theta}$.

4. Check the convergence criterion. If this criterion is not satisfied, increase the step number $n$ by one and go to stage 2.

In Eqs. (4.1), $\Phi'$ is the Frechet derivative of the operator $\Phi$; $(I_i)'_{\boldsymbol{u}}$ and $(I_i)''_{\boldsymbol{u}}$ are the first- and second-order Frechet derivatives of $I_i$ with respect to $\boldsymbol{u}$. In particular, $\langle (I_1)'_{\boldsymbol{u}}(\boldsymbol{u}_n), \boldsymbol{w}\rangle = 2(\boldsymbol{u}_n - \boldsymbol{u}_d, \boldsymbol{w})_Q$ and $(I_1)''_{\boldsymbol{u}}(\boldsymbol{u}_n)(\tilde{\boldsymbol{u}}, \boldsymbol{w}) = 2(\tilde{\boldsymbol{u}}, \boldsymbol{w})_Q$.

In our computational experiments, the convergence criterion was the inequality $\|T_{n+1} - T_n\|/\|T_n\| < 10^{-6}$. We used the open source software freeFEM++ to solve system (4.1) numerically by the finite element method.

Let us discuss the results of the computational experiments. First, we consider the problem of a plane viscous incompressible fluid flow around a circular cylinder $\Omega_1$ in a channel $\Omega_2$. Figure 1a shows the streamlines of the flow obtained by solving the two-dimensional boundary-value problem

$$-(1/\mathrm{Re})\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u} + \mathrm{grad}\, p = \tilde{\boldsymbol{G}}, \qquad \mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega = \Omega_2 \setminus \Omega_1; \tag{4.2}$$

$$\boldsymbol{u}\Big|_{\Gamma_0} = \boldsymbol{0}, \qquad \boldsymbol{u}\Big|_{\Gamma_1} = \boldsymbol{g}_1(y), \qquad \frac{1}{\mathrm{Re}}\frac{\partial\boldsymbol{u}}{\partial n} - p\boldsymbol{n}\Big|_{\Gamma_2} = \boldsymbol{0} \tag{4.3}$$

for dimensionless Navier–Stokes equations at $\Omega_2 = (0, 10) \times (0.3)$, $\Omega_1 = \{(x, y)\colon\ (x - 4.5)^2 + (y - 2)^2 < 0.25\}$, and Re = 100. The boundary conditions (4.3) correspond to the following conditions: 1) no-slip conditions on the rigid

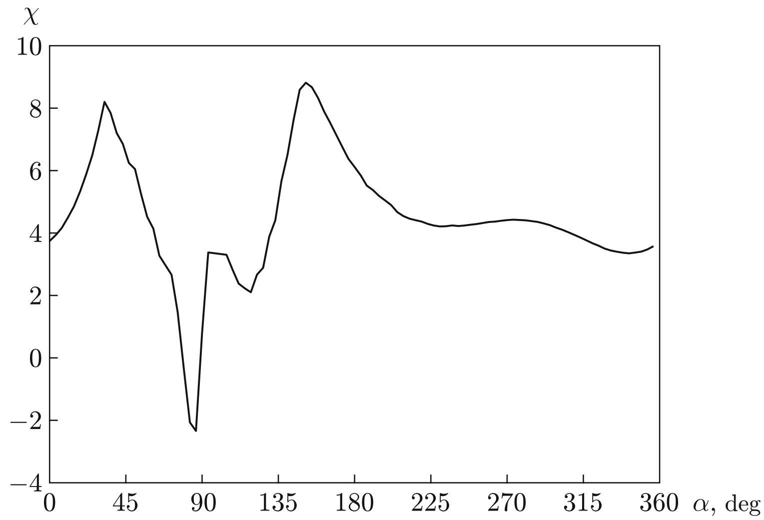


Fig. 2. Distribution of the dimensionless heat flux over the cylinder surface.

TABLE 1

Drag Coefficient $C_x$ for Cases Shown in Fig. 1 for Different Reynolds Numbers

| Re | $C_x$ | | |
|---|---|---|---|
| | Fig. 1a | Fig. 1b | Fig. 1c |
| 20 | 3.73 | 3.09 | 2.41 |
| 40 | 2.52 | 1.96 | 1.24 |
| 60 | 2.04 | 1.59 | 0.86 |
| 80 | 1.78 | 1.40 | 0.68 |
| 100 | 1.62 | 1.28 | 0.57 |

boundaries $\Gamma_0$ (cylinder surface and side walls of the channel); 2) setting a parabolic profile $\boldsymbol{g}_1(y) = \{y(3-y), 0\}$ for velocity on the inflow segment $\Gamma_1$; 3) "do nothing" boundary condition $(1/\operatorname{Re})\,\partial\boldsymbol{u}/\partial n - p\boldsymbol{n} = \boldsymbol{0}$ on the outflow segment $\Gamma_2$. The latter condition was used, in particular, in [10, 11] dealing with extremum problems for steady Navier–Stokes equations with the control being the function $\boldsymbol{g}$ in Eq. (1.3) or the density of external forces. It is seen in Fig. 1a that a vortex zone is formed behind the body. To reduce this zone, we solve an extremum problem of minimizing the functional $\tilde{I}_2(\boldsymbol{u}) \equiv \|\operatorname{rot}\boldsymbol{u}\|^2$ for a dimensionless analog of the full model (1.1), (1.2) under the boundary conditions (4.3) for velocity and the condition $T = \psi = 0$ on $\Gamma_D \equiv \Gamma_1$ for temperature. Figure 1b shows the streamlines for a flow obtained by solving this problem with the Reynolds number Re = 100 and the Rayleigh number $\mathrm{R} = 10^5$ in the case with the heat flux $\chi$ controlled through the body surface. Figure 1c shows the streamlines obtained by minimizing $\tilde{I}_2(\boldsymbol{u})$ with choosing the heat flux on the cylinder surface and on the nearby parts of the channel walls, which are indicated by the dashed curves.

Figure 2 shows the dimensionless heat flux $\chi$ through the cylinder surface as a function of the angular variable $\alpha \in [0, 360°]$ for the second variant of control, illustrated in Fig. 1c. The control results in fluid cooling on the upper surface of the cylinder; as a result, the fluid flows around the body in the downward direction. The fluid on the lower surface, vice versa, is heated and moves upward around the body; for this reason, flow separation does not occur.

An important characteristic of the flow around the body is the drag coefficient calculated by the formula $C_x = 2F_x/(DU^2)$, where $F_x$ is the horizontal component of the force acting on the body from the fluid, $D$ is the cylinder diameter, and $U$ is the characteristic velocity of the flow incoming onto the body. The values of the drag coefficient $C_x$ for the uncontrolled flow (see Fig. 1a) and for the first and second variants of control (see Figs. 1b and 1c) for different values of the Reynolds number Re are summarized in Table 1. An analysis of this table shows that the drag coefficient is substantially decreased by using the second variant of flow control (see Fig. 1c), as compared with the uncontrolled flow case, especially at Re = 80 and 100.

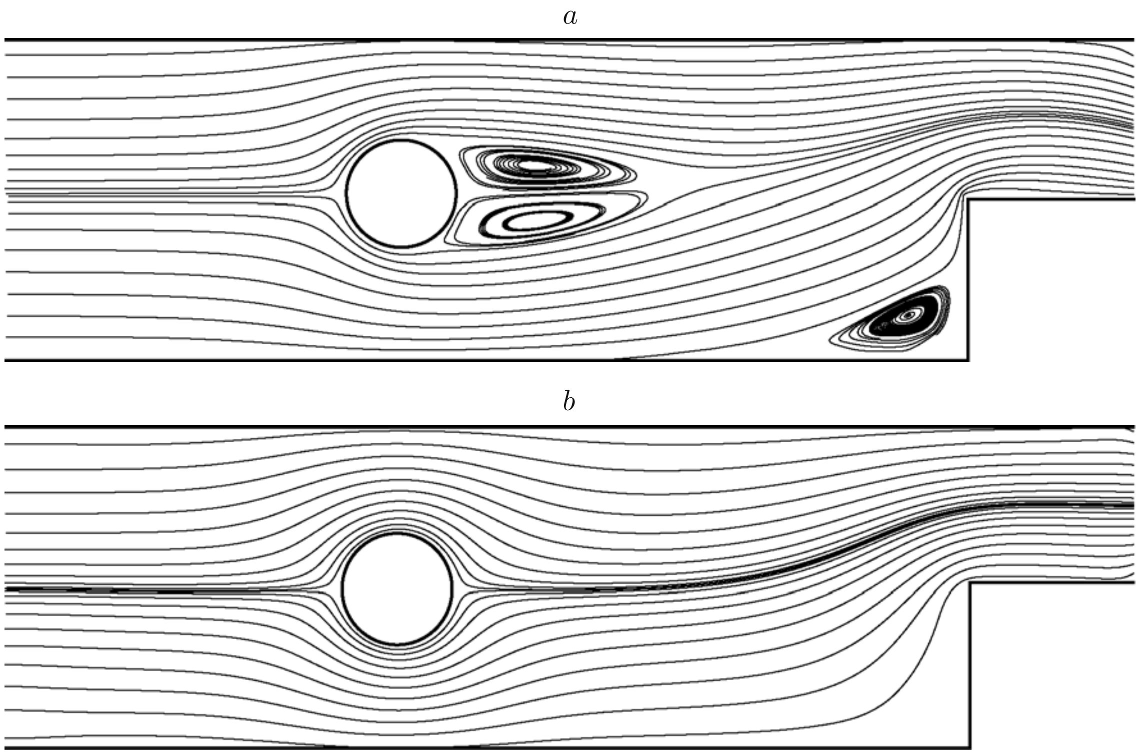


Fig. 3. Streamlines in the flow around a cylinder in a channel with a forward-facing step at Re = 100: (a) uncontrolled flow; (b) flow controlled through the body surface and the solid walls of the channel at $R = 10^5$.

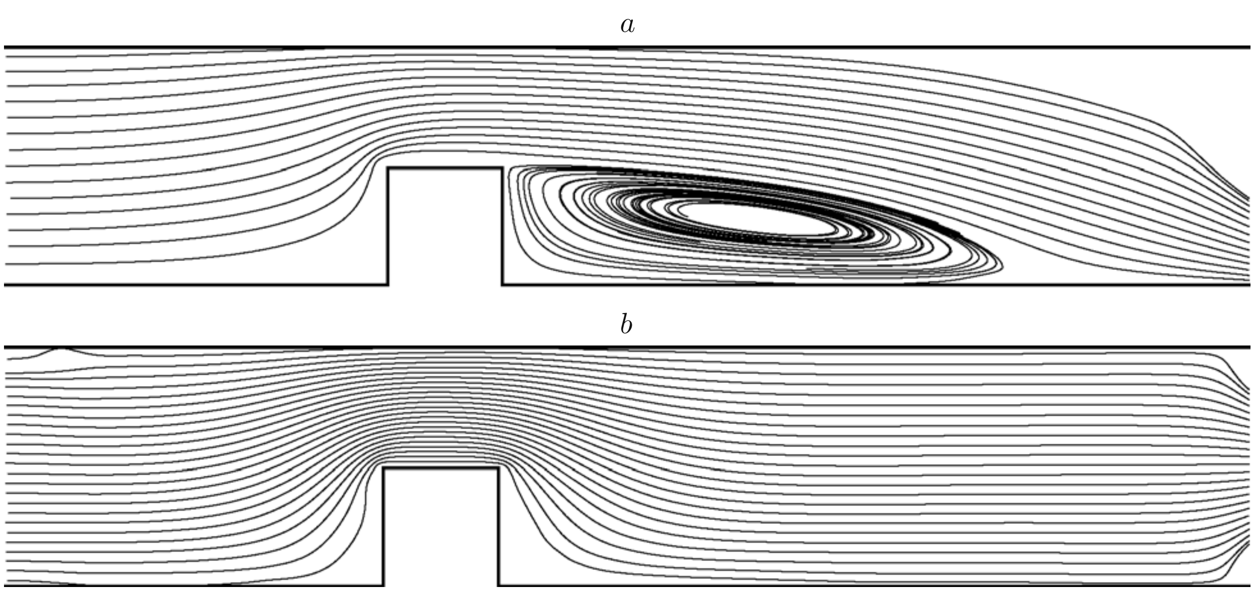


Fig. 4. Streamlines in a channel with a step at Re = 100: (a) uncontrolled flow; (b) flow controlled through the channel walls at $R = 10^5$.

Let us consider a similar formulation of the problem of the flow around a circular cylinder in a channel with a forward-facing step. Figure 3a shows the streamlines of the flow obtained by solving the two-dimensional boundary-value problem (4.2), (4.3) at Re = 100. It is seen that channel constriction generates an additional vortex zone in the right bottom corner. To reduce both vortex zones, we again solve the problem of minimizing the functional $\tilde{I}_2(\boldsymbol{u})$. Figure 3b shows the flow obtained by solving this problem at Re = 100 and R = $10^5$ with the control by choosing the heat flux $\chi$ on the cylinder surface and on the solid walls of the channel. It is seen that the vortex zones can be substantially reduced, and the structure of the resultant flow is similar to the vortex-free viscous fluid flow.

Let us consider the flow in a channel with a step. Figure 4a shows the streamlines constructed by solving the two-dimensional boundary-value problem (4.2), (4.3) at Re = 100. It is seen that a vortex zone is formed behind the step. To reduce this zone, we solve the problem of minimizing the functional $\tilde{I}_2(\boldsymbol{u})$. Figure 4b shows the flow obtained by solving this problem at Re = 100 and R = $10^5$ with controlling the heat flux $\chi$ on the channel walls. The structure of the resultant flow is similar to the vortex-free viscous fluid flow.

An analysis of the results obtained and their comparisons with the data [10, 11] show that the use of the boundary temperature control allows one to affect the velocity field of viscous fluid flows and to obtain flows with prescribed dynamic properties.

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